

Digital control

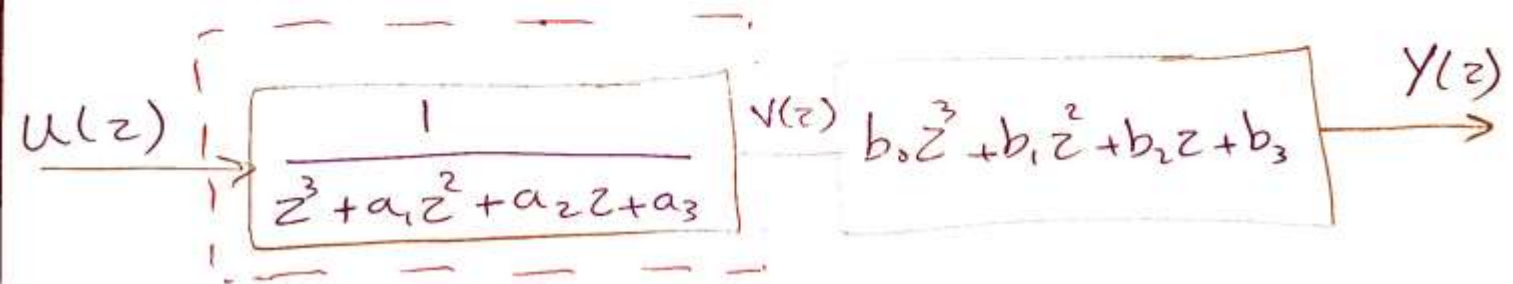
$$T.F = \frac{Z^3 + 2Z + 3}{Z^3 + 4Z^2 + 2Z - 1}$$

→ Find the state-space model in controllable and observable form.

→ Draw state diagram for each form

*3rd order system

$$T.F = \frac{Y(z)}{U(z)} = \frac{b_0 Z^3 + b_1 Z^2 + b_2 Z + b_3}{Z^3 + a_1 Z^2 + a_2 Z + a_3}$$



$$\frac{V(z)}{U(z)} = \frac{1}{Z^3 + a_1 Z^2 + a_2 Z + a_3}$$

$$U(z) = (Z^3 + a_1 Z^2 + a_2 Z + a_3) V(z) \Big| Z^{-1} \cdot T$$

$$U(k) = V(k+3) + a_1 V(k+2) + a_2 V(k+1) + a_3 V(k) \quad \text{--- (1)}$$

$$x_1(k) = v(k)$$

$$x_2(k) = v(k+1) = x_1(k+1)$$

$$x_3(k) = v(k+2) = x_2(k+1)$$

$$x_3(k+1) = v(k+3)$$

In Continuous

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

In discrete

$$x(k+1)T = Ax(kT) + Bu(kT)$$

$$y(kT) = Cx(kT) + Du(kT)$$

From ①

$$v(k+3) = -a_3 v(k) - a_2 v(k+1) - a_1 v(k+2) + u(k)$$

$$x_3(k+1) = -a_3 x_1(k) - a_2 x_2(k) - a_1 x_3(k) + u(k)$$

$$\begin{bmatrix} x_1(k+1) \\ x_2(k+1) \\ x_3(k+1) \end{bmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

← البسط
المقام
فيجب المعادلة الثانية.

$$\frac{Y(z)}{V(z)} = b_0 z^3 + b_1 z^2 + b_2 z + b_3$$

$$Y(z) = (b_0 z^3 + b_1 z^2 + b_2 z + b_3) V(z) \quad \downarrow z^{-1} \cdot T$$

$$y(k) = b_0 v(k+3) + b_1 v(k+2) + b_2 v(k+1) + b_3 v(k)$$

$$y(k) = b_0 x_3(k+1) + b_1 x_3(k) + b_2 x_2(k) + b_3 x_1(k)$$

$$= b_0 (-a_3 x_1(k) - a_2 x_2(k) - a_1 x_3(k) + u(k))$$

$$+ b_1 x_3(k) + b_2 x_2(k) + b_3 x_1(k)$$

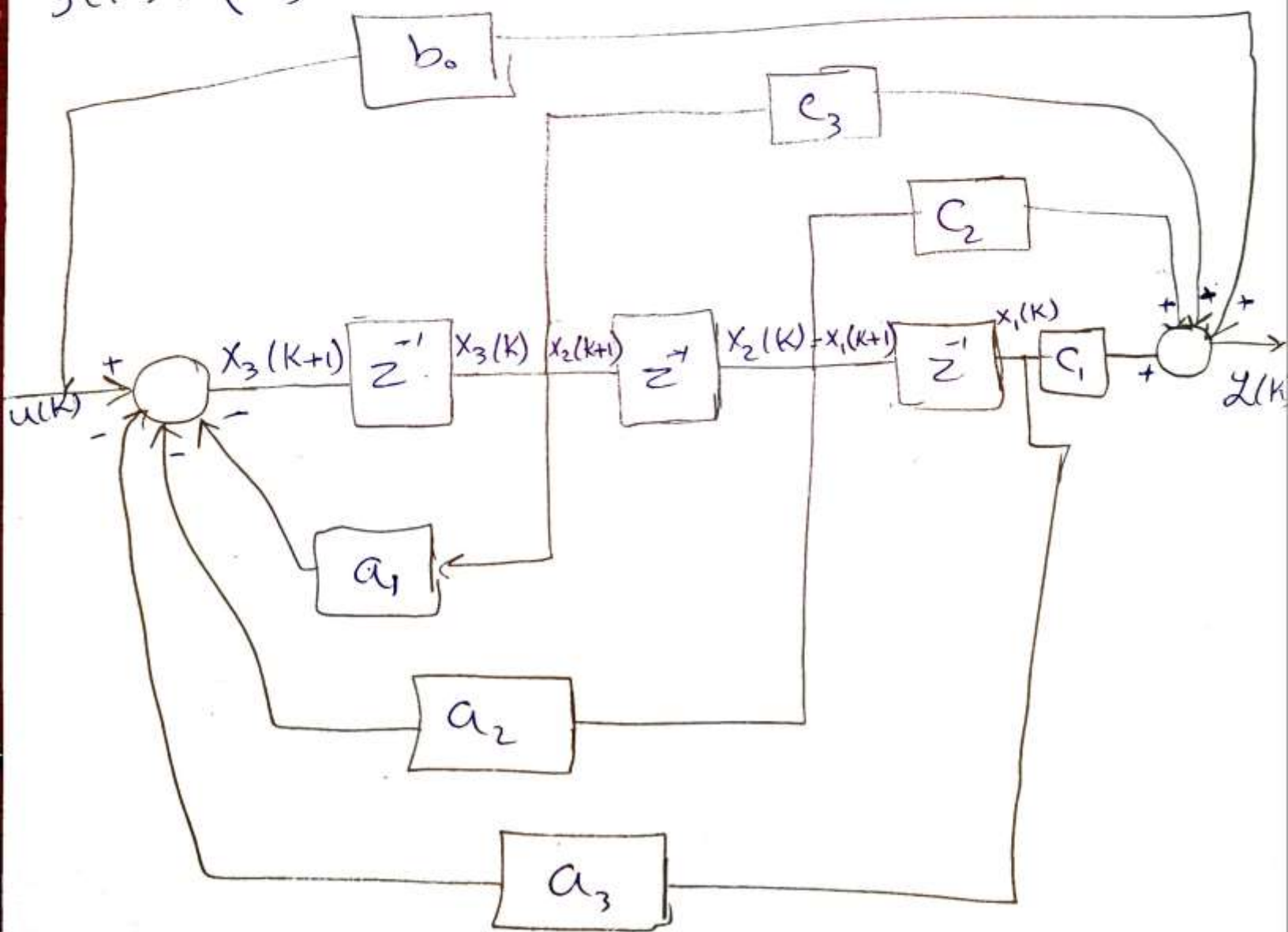
$$y(k) = (b_3 - b_0 a_3) x_1(k) + (b_2 - a_2) x_2(k) +$$

$$(b_1 - a_1) x_3(k) + b_0 u(k)$$

$$y(k) = \underbrace{(b_3 - b_0 a_3)}_{c_1} \quad \underbrace{b_2 - b_0 a_2}_{c_2} \quad \underbrace{b_1 - b_0 a_1}_{c_3} \begin{bmatrix} x_1(k) \\ x_2(k) \\ x_3(k) \end{bmatrix} + b_0 u(k)$$

if $b_0 = 0$

$$y(k) = (b_3 \quad b_2 \quad b_1) x(k)$$



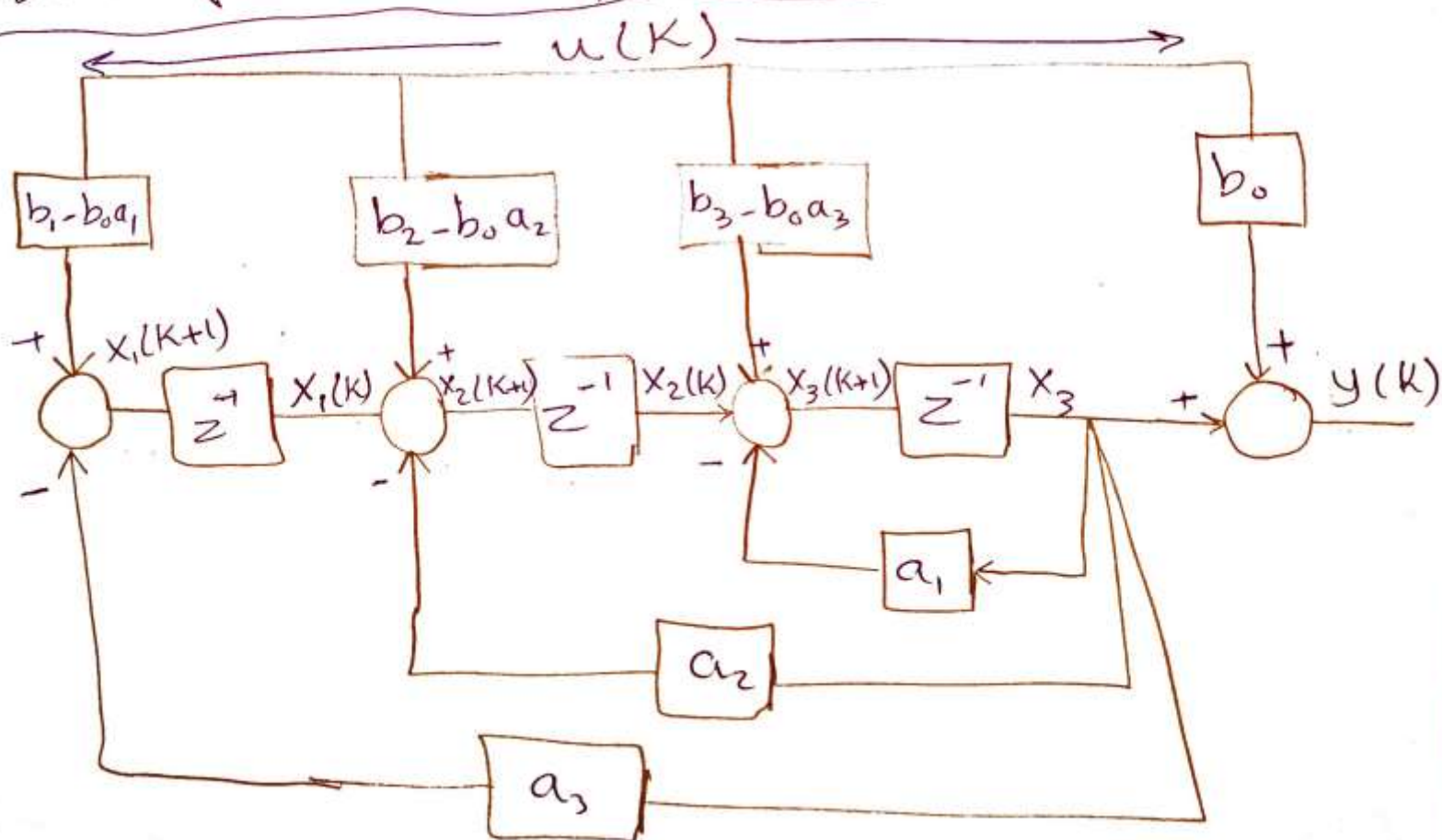
* observable form:

$$A_o = A_c^T \quad ; \quad C_o = B_c^T$$

$$B_o = C_c^T \quad ; \quad D_o = D_c = b_o$$

$$X(K+1) = \begin{pmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{pmatrix} X(K) + \begin{pmatrix} b_3 - b_o a_3 \\ b_2 - b_o a_2 \\ b_1 - b_o a_1 \end{pmatrix} u(K)$$

$$y(K) = (0 \quad 0 \quad 1) X(K) + b_o u(K)$$



EX

$$T.F = \frac{z^3 + 2z + 3}{z^3 + 4z^2 + 2z - 1}$$

Controllable Form:-

$$x(k+1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & -4 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 4 & 0 & -4 \end{pmatrix} x(k) + \begin{pmatrix} 1 \end{pmatrix} u(k)$$

\swarrow \downarrow \downarrow \uparrow
 $3 - 1(-1)$ $2 - 1(2)$ $0 - 1(-4)$ $b_0 = 1$

observable Form

$$x(k+1) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & -4 \end{pmatrix} x(k) + \begin{pmatrix} 4 \\ 0 \\ -4 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} x(k) + u(k)$$

Diagonal Form

Ex

$$T.F = \frac{2z^3 + z + 1}{z(z+0.5)(z-0.5)}$$

$\hookrightarrow z(z^2 - 0.25)$

$$\begin{array}{r} 2 \\ \hline 2z^3 + z + 1 \\ z^3 - 0.25z \quad -2z^3 \quad +0.5z \\ \hline 1.5z + 1 \end{array}$$

$$T.F = \frac{1.5z + 1}{z(z+0.5)(z-0.5)} + 2$$

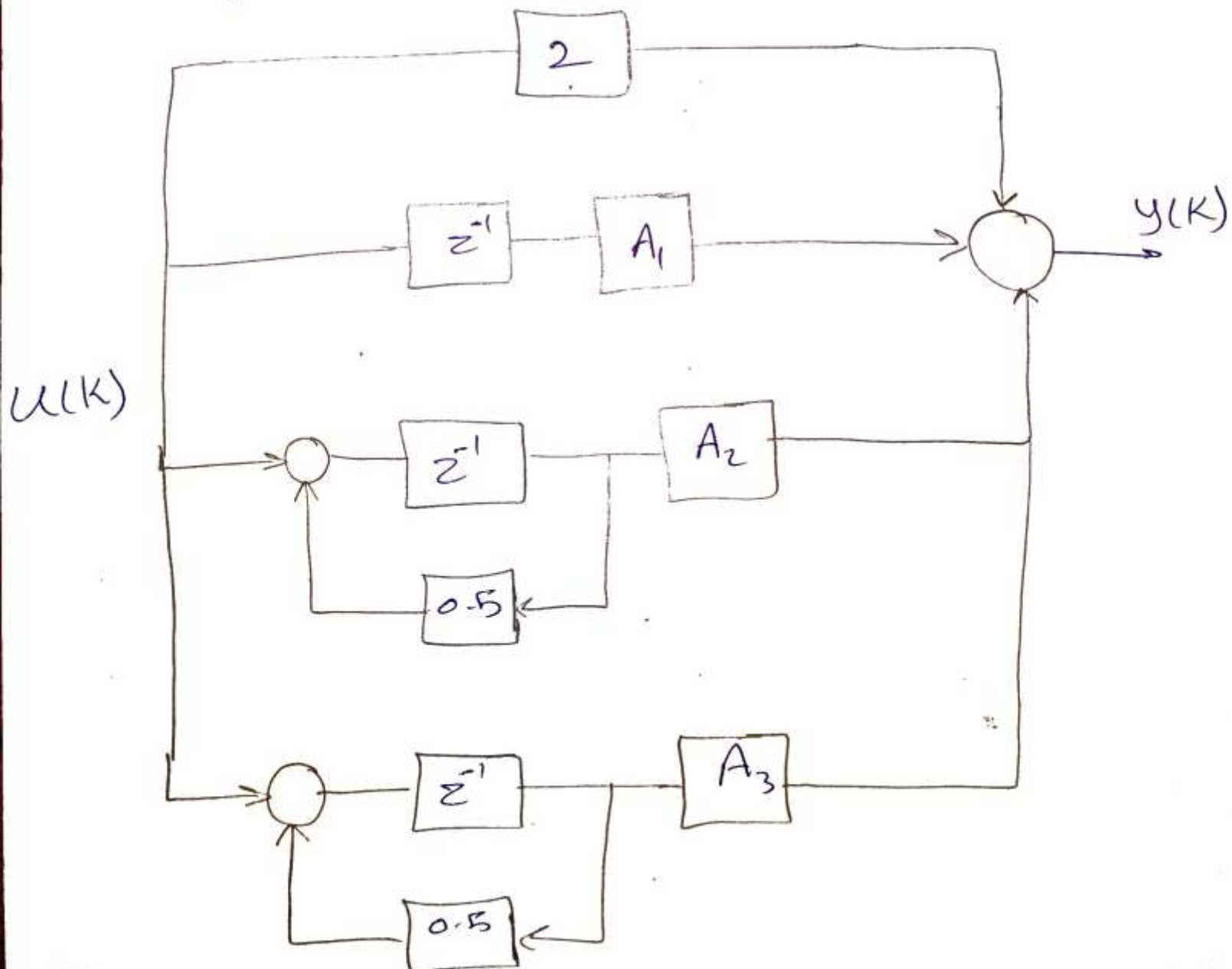
using P.F

$$= 2 + \frac{A_1}{z} + \frac{A_2}{z+0.5} + \frac{A_3}{z-0.5}$$

$$T.F = 2 - \frac{4}{z} - \frac{1}{z+0.5} - \frac{1}{z-0.5}$$

$$X(k+1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -0.5 & 0 \\ 0 & 0 & -0.5 \end{pmatrix} X(k) + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = (A_1 \quad A_2 \quad A_3) X(k) + 2 u(k)$$



State-Space Analysis

Given

$$x(k+1) = A x(k) + B u(k)$$

$$y(k) = C x(k) + D u(k)$$

$A, B, C, D \rightarrow$ given

Q1 T.F

$$x(k+1) = A x(k) + B u(k) \rightarrow Z x(z)$$

$$z x(z) = A x(z) + B u(z)$$

$$z x(z) - A x(z) = B u(z)$$

$$(zI - A) x(z) = B u(z)$$

$$x(z) = B (zI - A)^{-1} u(z) \rightarrow \textcircled{1}$$

$$y(k) = C x(k) + D u(k)$$

$$y(z) = C x(z) + D u(z) \rightarrow \textcircled{2}$$

From ① in ②

$$Y(z) = C(zI - A)^{-1} B u(z) + D u(z) \\ = (C(zI - A)^{-1} B + D) u(z)$$

$$\boxed{T.F = \frac{Y(z)}{U(z)} = C(zI - A)^{-1} B + D}$$

② ch. equation

$$|zI - A| = 0$$

$$T.F = \frac{\text{مخرج}}{\text{مدخل}}$$

له مقدار = مقدار

③ check the system controllability & observability:-

a) Controllability

~ The system is controllable if for any change of an external input, produce change in internal states of the system.

observability:

The system is observable, if we can determine or estimate the states' values from the relation between input and output or by the history information. For the o/p and i/p.

→ controllability matrix (M_c):

$$M_c = (B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B) \quad n \rightarrow \text{system order.}$$

2nd order $M_c = (B \quad AB)$

3rd order $M_c = (B \quad AB \quad A^2B)$

if $|M_c| \neq 0 \rightarrow \text{system is controllable.}$

* observability matrix (M_o)

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{pmatrix}$$

$$\text{if } |M_o| \neq 0$$

→ system is observable.

2nd order

$$M_o = \begin{pmatrix} C \\ CA \end{pmatrix}$$

3rd order

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

4) system response

$$\text{Let } \phi(z) = (zI - A)^{-1}$$

$$x(z) = \phi(z) \cdot z x(0) + \phi(z) B u(z)$$

$$\downarrow z^{-1} \cdot T$$

$$x(k) = \dots$$

$$\text{System response: } y(k) = C x(k) + D u(k)$$

Ex

$$x(k+1) = \begin{pmatrix} 0 & 0.6321 \\ -1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = (1 \quad -0.6321) x(k)$$

① Find T.F

② check stability

$$\underline{T.F} = C (zI - A)^{-1} B + D$$

$$(zI - A) = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0 & 0.6321 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} z & -0.6321 \\ 1 & z-1 \end{pmatrix}$$

$$(zI - A)^{-1} = \frac{1}{z(z-1) + 0.6321} \begin{pmatrix} z-1 & 0.6321 \\ -1 & z \end{pmatrix}$$

$$T.F = (1 \quad -0.6321) \frac{1}{(z-1)z + 0.6321} \begin{pmatrix} z-1 & 0.6321 \\ -1 & z \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{z(z-1) + 0.6321} (1 \quad -0.6321) \begin{pmatrix} z-1 + 0.6321 \\ -1 + z \end{pmatrix}$$

$$T.F = \frac{z^{-1} + 0.6321 + 0.6321 - 0.6321 z}{z^2 - z + 0.6321}$$

$$T.F = \frac{0.3679z + 0.2642}{z^2 - z + 0.6321}$$

ch. eqn $z^2 - z + 0.6321 = 0$

$$Z_{1,2} = 0.5 \pm j0.618$$

Poles

$$|Z_{1,2}| = 0.795 < 1 \rightarrow \text{system stable}$$

Ex

$$x(k+1) = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u(k)$$

$$y(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(k)$$

if $x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$; find unit step response.

$$X(z) = \phi(z) z x(0) + \phi(z) B u(z)$$

$$\phi(z) = (zI - A)^{-1}$$

$$zI - A = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} - \begin{pmatrix} 0.8 & 0 \\ 0 & 0.2 \end{pmatrix} = \begin{pmatrix} z-0.8 & 0 \\ 0 & z-0.2 \end{pmatrix}$$

$$(zI - A)^{-1} = \frac{1}{(z-0.8)(z-0.2)} \begin{pmatrix} z-0.2 & 0 \\ 0 & z-0.8 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} = (zI - A)^{-1} = \phi(z)$$

$$X(z) = \begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} * z * \begin{pmatrix} 1 \\ 1 \end{pmatrix} +$$

$$\begin{pmatrix} \frac{1}{z-0.8} & 0 \\ 0 & \frac{1}{z-0.2} \end{pmatrix} * \begin{pmatrix} 1 \\ 1 \end{pmatrix} * \left(\frac{z}{z-1} \right)$$

unit step

$$X(z) = \begin{bmatrix} \frac{z}{z-0.8} \\ \frac{z}{z-0.2} \end{bmatrix} + \begin{bmatrix} \frac{1}{z-0.8} \\ \frac{1}{z-0.2} \end{bmatrix} \frac{z}{z-1}$$

$$= \begin{bmatrix} \frac{z}{z-0.8} \\ \frac{z}{z-0.2} \end{bmatrix} + \begin{bmatrix} \frac{z}{(z-0.8)(z-0.2)} \\ \frac{z}{(z-0.2)(z-1)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{z}{z-0.8} \\ \frac{z}{z-0.2} \end{bmatrix} + \begin{bmatrix} \frac{-1.5z}{z-0.8} + \frac{5z}{z-1} \\ \frac{-1.25z}{z-0.2} + \frac{1.25z}{z-1} \end{bmatrix}$$

مخرجنا الـ z
 به لحذف ما نجيب
 الـ Partial
 fraction

$$\underline{z^{-1} \cdot T}$$

$$X(k) = \begin{bmatrix} (0.8)^k \\ (0.2)^k \end{bmatrix} + \begin{pmatrix} -5(0.8)^k + 5 \\ -1.25(0.2)^k + 1.25 \end{pmatrix}$$

$$X(K) = \begin{pmatrix} (0.8)^K - 5(0.8)^K + 5 \\ (0.2)^K - 1.25(0.2)^K + 1.25 \end{pmatrix}$$

$$= \begin{pmatrix} -4(0.8)^K + 5 \\ -0.25(0.2)^K + 1.25 \end{pmatrix}$$

→ unit step response

$$Y(K) = (1 \quad 1) \begin{pmatrix} -4(0.8)^K + 5 \\ -0.25(0.2)^K + 1.25 \end{pmatrix}$$

$$= -4(0.8)^K + 5 - 0.25(0.2)^K + 1.25$$

$$Y(K) = -4(0.8)^K - 0.25(0.2)^K + 6.25$$

Ex

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}; B = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}; C = (1 \ 0 \ 2)$$

check system Controllability and observability

$$M_c = (B \quad AB \quad A^2 B)$$

$$AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix}$$

$$A^2 B = A \cdot AB = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} 3 \\ 11 \\ -5 \end{pmatrix} = \begin{pmatrix} -2 \\ 7 \\ -20 \end{pmatrix}$$

$$M_c = \begin{bmatrix} 0 & 3 & -2 \\ 1 & 11 & 7 \\ 3 & -5 & -20 \end{bmatrix}$$

$$|M_c|_{s=1} \begin{vmatrix} 3 & -2 \\ -5 & -20 \end{vmatrix} + 3 \begin{vmatrix} 3 & -2 \\ 11 & 7 \end{vmatrix}$$

$$= 70 + 129 = 199 \neq 0 \rightarrow \text{system controllable}$$

$$M_o = \begin{pmatrix} C \\ CA \\ CA^2 \end{pmatrix}$$

$$CA_s(1 \ 0 \ 2) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix} = (-1 \ -4 \ -1)$$

$$CA^2_s CA_s A_s (-1 \ -4 \ -1) \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \\ -1 & -2 & -1 \end{pmatrix}$$

$$= (0 \ -6 \ -12)$$

$$M_o = \begin{pmatrix} 1 & 0 & 2 \\ -1 & -4 & -1 \\ 0 & -6 & -12 \end{pmatrix}$$

$$|M_o| = 1 \begin{vmatrix} -4 & -1 \\ -6 & -12 \end{vmatrix} + 1 \begin{vmatrix} 0 & 2 \\ -6 & -12 \end{vmatrix} = 54 \neq 0 \rightarrow \text{system observable}$$